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An algorithm for the complete symmetry classification of differential equations based on Wu's method

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Abstract In this paper, an alternative algorithm which uses Wu's method (differential characteristic set algorithm) for the complete symmetry classification of (partial) differential equations containing arbitrary parameter is proposed. The classification is determined by decomposing the solution set of determining equations into a union of a series of zero sets of differential characteristic sets of the corresponding differential polynomial system of the determining equations. Each branch of the decomposition yields a class of symmetries and associated parameters. The algorithm makes the classification become direct and systematic. This is also a new application of Wu's method in the field of differential equations. As illustrative examples of our algorithm, the complete potential symmetry classifications of linear and nonlinear wave equations with an arbitrary function parameter and both classical and nonclassical symmetries of a parametric Burgers equation are presented.

Keywords Differential characteristic set algorithm · Parametric differential equations · Symmetry classification · Wu's method

1 Introduction

1.1 Symmetry-classification problems

The classical symmetry method, originally developed by the Norwegian mathematician Sophus Lie (1842–1899), leads us to one-parameter groups of transformations acting on the space of independent and dependent variables that leave the considered (partial) differential equations (PDEs) unchanged. In his work, Lie pointed out that this type of symmetry group is of great importance to understand and construct solutions of PDEs. Nowadays, Lie's theory has been widely used in diverse fields of mathematics and in almost any area of theoretical physics [1, Chap. 1], [2, Chap. 4].

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In symmetry analysis of PDEs, traditionally, there are two interrelated problems. The first one is finding the maximal symmetry admitted by a given system of PDEs. The principal tool for handling this is the classical infinitesimal algorithm (Lie's algorithm) developed by Lie. It reduces the problem to finding the corresponding Lie algebra of infinitesimal vectors (InfVs) whose infinitesimal functions are found as solutions of some over-determined system of PDEs, called determining equations (DTEs) [1, Sect. 2.4], [2, Sect. 4.1.3], [3, Chap. 2]. In determining symmetries, tedious, mechanical computations are involved. Even the determining of symmetries of a modest PDE is prone to fail, if done with pencil and paper. Programmable computer-algebra systems (CASs) such as Mathematica, Maple, are extremely useful aids in such computations. Reid [4–7], Schwarz [8,9], Wolf and Brand [10], Mansfield [11,12], Lisle and Reid [13], Boulier et al. [14,15] and many others (see [16, Chap. 13], [17] and references therein) partially implemented algorithms and sophisticated symbolic codes in CASs for that purpose. Of them, the CAS Maple packages `rifsimp`, `diffalg` and `diffgrob2` are more powerful and widely used. Cartan's exterior-form approach [18], Janet–Riquier theory [8,9,19,20], Gröbner base algorithm [11], Rosenfeld–Gröbner algorithm [14], Ritt–Kolchin differential-algebra method [21,22], formal power-series analysis [6,7], etc can be used to solve DTEs. However, there are still many open problems in determining symmetries [23]. The second problem is classifying PDEs that admit a prescribed symmetry group. For a family of PDEs with arbitrary parameters θ, finding both the parameters θ and corresponding maximal set of symmetries G_θ is called the symmetry-classification problem of the family of PDEs [3, pp. 61–68]. The problem is not only interesting from a purely mathematical point of view, but also important for practical applications. A variety of PDEs recognized in engineering and physical science as mathematical models for a diversity of natural phenomena involve arbitrary parameters or constitutive laws. Naturally, these arbitrary elements are determined experimentally or from a "simplicity criterion." It often occurs that one can achieve the same result by the requirement that an arbitrary element be such that the corresponding model equation admits an additional symmetry. Thus we come to the problem of symmetry classification of PDEs. For example, the wave equation $u_{tt} = (F(u)u_x)_x$ and the Burgers–KdV equation $u_t + \alpha u u_x + \beta u_{xx} + \gamma u_{xxx} = 0$ contain parameters $\theta = \{F(u)\}$ and $\theta = \{\alpha, \beta, \gamma\}$. The symmetries G_θ of these equations are different as different values are taken by the parameters θ [24]. The symmetry classification permits us to choose purposefully the proper form and correct values of such parameters θ so that we get the best modeling of physics problems. This point of view is supported by the fact that the most successful mathematical models in theoretical and applied science have a rich symmetry structure. Indeed, the basic equations of modern physics, the wave, Schrödinger, Dirac and Maxwell equations are distinguished from the whole set of PDEs by their Lie and non-Lie (hidden) symmetries; see, e.g., [25, Chap. 2] for more details on symmetry properties of these equations.

The first systematic investigation of the problem of symmetry classification was carried out by Lie [26] for linear second-order PDEs with two independent variables. Today, there is a considerable literature on the symmetry-classification of PDEs of physical interest. Problems of general symmetry classification, except for really trivial cases, are very difficult. Apart from the difficulties that exist in determining symmetries, we have additional difficulties in symmetry-classification problems brought about by involved parameters. For this reason, finding an effective approach to simplification is essentially needed. There are different techniques used to solve symmetry-classification problems. An accepted common strategy is the utilization of equivalence transformations. The equivalence relation divides the set of all PDEs of a given family into disjoint classes of equivalent equations. One chooses a representative for each of the classes, thus simplifying the DTEs. The method, which is quite intricate in general, is efficient when applied to particular families of PDEs. The second method applied to the problem is the method of preliminary symmetry classification [27]. One can observe in applications of symmetry analysis that most symmetries resulting from symmetry classification are, in effect, subgroups of equivalence groups. Accordingly, one can consider the restricted problem of symmetry classification by taking symmetries from equivalence transformations. Then the problem reduces to the construction of optimal systems of Lie subalgebras. Lie [26] pursued this way in his classification of ordinary differential equations. This approach was also employed by Akhatov et al. [27–29] and was called the method of preliminary group classification. Recently, Zhdanov et al. [30,31] developed a compatibility method for solving the symmetry-classification problem for the nonlinear Schrödinger equation. In [32,33], Popovych et al. extended the method to the complete symmetry classification of an evolution equation by further considering the so called additional equivalence transformations.

In compatibility methods, it seems that inspecting specific classifying equations (or relations) satisfied by parameters from DTEs is a critical point. In the preliminary method, finding the optimal systems of Lie subalgebras is a hard task. Especially, in higher-dimensional and nonlinear cases, it is too difficult to get these systems.

On implementation of symbolic computation codes, the Maple packages `rifsimp` and `diffalg` can be used to perform the complete symmetry classifications [4–7,15,34]. When applied to symmetry classifications, they yield the classification trees, automatically executed by both `diffalg` and `rifsimp` in Maple.

1.2 Characteristic-set method

In mechanical (automatic) theorem-proving fields, there is a fundamental algorithm called the characteristic-set algorithm (also named Wu's method) [35], [36, Chap. 3], [37], established by the Chinese mathematician Wu Wen Tsun in the 1970s, based on Ritt's theory [22, Chap. 5]. It also has become a fundamental algorithmic theory in algebraic geometry together with the Gröbner base algorithm [38, Chap. 2]. The method has been applied in a wide range of science fields, such as mechanical theorem proving [35], optimization problems, surface-fitting problems in CAGD, Bar Linkage Design, $\cdots$, etc [36, Chap. 36]. The differential analogue of Wu's method was proposed in the 1980s [37]. The method is more especially on target to deal with the zero set of a differential polynomial system (dps) and efficient differential elimination without directly involving the concept of an algebra ideal. The analysis of zero sets of a dps in the method gives rise to the fundamental notion of a differential characteristic set (dchar-set) and further principles under the names: Well-ordering principles, Zero-decomposition theorems, Variety-decomposition theorems, etc.

In Wu's algorithmic theory, a geometry Theorem is defined as

Definition 1 A theorem consists of a dps called a hypothesis set (denote it as HYP) and a dps called a conclusion set (denote it as CONC). Then we say that the theorem is **TRUE** if $\mathrm{Zero(HYP)} \subseteq \mathrm{Zero(CONC)}$, i.e.,

$$\mathrm{CONC}\big|_{\mathrm{HYP}=0} = 0. \tag{1}$$

On mechanical theorem proving, Wu gave the following fundamental Theorem [37].

Theorem 1 (Principle of mechanical theorem proving) *For a Theorem with hypothesis set* HYP *and conclusion set* CONC*, if* DCS *is a dchar-set of* HYP *and* $\mathrm{Prem(CONC/DCS)} = 0$*, then the theorem is true under the non-degenerate conditions* $\mathrm{IS} \neq 0$*, where* IS *is an* ISP *of the* DCS.

The definitions of dchar-set, remainder operator Prem and ISP of a dps, etc are given in the next section. Because the operator Prem is completely constructive, a Theorem can be proved by computer using Theorem 2.

In usual geometry theorems, the geometric entities and geometric relations are understood to be in the usual exact geometrical sense so that degeneracies are implicitly discarded. Otherwise the theorem may not be true or even devoid of meaning. In Theorem 2, IS is an important object. In geometry, $\mathrm{IS} = 0$ corresponds to the degeneracy cases of the geometric figure. While, in solving algebraic equations, it represents the solvable conditions of $\mathrm{HYP} = 0$. The dchar-set algorithm used in the theorem gives a way to get these degenerate cases and check whether the theorem is true on these degenerate cases.

Although the two subjects (theorem proving and symmetry classification) are far from each other, the formal similarity statements of both theorem proving and symmetry criterion (compare (1) and (3)) enlighten us that the method used in mechanical theorem proving can be used to deal with the symmetry-classification problem. We will show that Wu's method is more suitable to deal with symmetry-classification problems. And the classifying equations come from the degeneracy conditions of dchar-sets of the dps obtained from DTEs. This is the critical step toward obtaining the complete symmetry classifications of a given PDE system.

1.3 Contents of the paper

In this article, we look into the symmetry-classification problem of a class of PDEs with arbitrary parameters from a different point of view by employing the differential form of Wu's method. We will show that Wu's method provides us with a direct, systematic algorithm for determining the complete symmetry classification of a class of PDEs. Therefore, the algorithm will lead to some explicit applications in physics and engineering.

The rest of this paper is organized as follows. In Sect. 2, we give a formal statement of the problem of the complete symmetry classification of parametric PDEs; in Sect. 3, we first briefly recall the dchar-set algorithm for a dps and basic results on it. Then, based on the dchar-set algorithm for a dps, an alternative algorithm for the complete symmetry classification of a system of PDEs is given; in Sect. 4, we present illustrative examples to show the efficiency of our algorithm; in Sect. 5, we give some concluding remarks and discussions.

2 The complete symmetry-classification problem

In this section, we give a mathematical description of the problem of the complete symmetry classification of a parametric family of PDEs.

Let $X = (x_1, x_2, \ldots, x_p)$ be independent variables and $U = (u_1, u_2, \ldots, u_q)$ be differentiable real functions of X. Use $U^\alpha = \{u_i^\alpha, i = 1, 2, \ldots, q\}$, where $\alpha = \{\alpha_1, \ldots, \alpha_p\} \in Z_+^p$($Z_+$ is the set of non-negative integers) and $u_i^\alpha = \frac{\partial^{|\alpha|} u_i}{\partial x_1^{\alpha_1} \ldots \partial x_p^{\alpha_p}}$, $(|\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_p)$, denote the derivative terms of U with respect to (w.r.t) X. Let $\partial U = \{U^\alpha, \alpha \in Z_+^p\}$. The notation $\mathcal{K}_X$ represents a differential field of functions of X with derivative operators $\partial_{x_i}, i = 1, 2, \ldots, p$ and $\mathcal{K}_X[\partial U]$ is the differential polynomial (d-pol) ring with indeterminates ∂U over $\mathcal{K}_X$. As usual in differential algebra, we use the notation Zero(DPS) for the zero points set (differential algebraic variety) of a dps DPS $\subset \mathcal{K}_X[\partial U]$ over a universal field of $\mathcal{K}_X$. This corresponds to the solution set of the DPEs DPS $= 0$. For a d-pol $I \in \mathcal{K}_X[\partial U]$, we use Zero(DPS$/I$) to denote the zero points of DPS with $I \neq 0$, i.e., Zero(DPS$/I$) = Zero(DPS)$\backslash$Zero(I) (See these preliminary concepts and notations in abstract differential algebra in [21, Chap. 4], [35], [36, Chap. 3], [37]).

Remark For the computation, $\mathcal{K}_X$ should be considered as a more concrete differential field, such as the differential field of meromophic functions or its various differential subfields.

We introduce the symmetry-classification problem by considering a family of PDEs

$$\Delta(\theta; X, \partial U) = 0, \tag{2}$$

with parameter θ. Without loss of generality, we consider both the case of parameter $\theta \in \mathcal{K}_X^m$ for an integer $m > 0$ (or we consider the problem on a proper extended field) and the case of $\Delta(\theta; X, \partial U) \subset \mathcal{K}_X[\partial U]$. Let G_θ be the maximal symmetry of the family of PDEs (2). The intersection of symmetry G_θ for all parameters θ will be called the kernel of symmetry G_θ, denoted by G_0 [3, pp. 66–67]. Because of this definition, the symmetry G_0 is admitted by the family of PDEs (2) for arbitrary parameter θ. It is equivalent to $G_0 \subseteq G_\theta$ for all parameters θ as G_0 and G_θ are sets of transformations. If for some particular value $\widetilde{\theta}$ of the parameter θ, $G_{\widetilde{\theta}} \neq G_0$, then the particular symmetry $G_{\widetilde{\theta}}$ corresponding to the value $\widetilde{\theta}$ is called an extension of G_0, and $\widetilde{\theta}$ can be called a specialization of the parameter θ. According to Lie's symmetry theory, we know that determining G_θ is equivalent to determining its InfV

$$\mathcal{X}_\theta = \vec{\xi}_\theta(X, U) \cdot \partial_x + \vec{\phi}_\theta(X, U) \cdot \partial_U.$$

The Inf$V$ corresponding to the kernel G_0 is denoted by $\mathcal{X}_0$, called the kernel algebra. For each specialization $\widetilde{\theta}$ of parameter θ, the Inf$V$ $\mathcal{X}_{\widetilde{\theta}}$ constitutes the Lie algebra of the family of PDEs (2) with $\theta = \widetilde{\theta}$. Hence, if $\mathcal{X}_{\widetilde{\theta}} \neq \mathcal{X}_0$, then $\mathcal{X}_{\widetilde{\theta}}$ is an extension of $\mathcal{X}_0$.

Now we describe the problem of the complete symmetry classification of a family of PDEs.

The problem of the complete symmetry classification: for a class of PDEs (2) with parameter θ, find the kernel G_0 of the maximal symmetry G_θ, and all specializations $\widetilde{\theta}$ of the parameter θ, viz. equivalently, determine $\mathcal{X}_0$ and all of its extensions $\mathcal{X}_{\widetilde{\theta}}$.

We use $\mathrm{Pr}\mathcal{X}_\theta$ to denote the prolongation vector of $\mathcal{X}_\theta$ on the space of $X \times \partial U$. Then, we have

Theorem 2 (Lie's criterion [1, pp. 161–162], [2, pp. 164–165]) *For the family of PDEs* (2) *with maximal rank, the operator* $\mathcal{X}_\theta$ *is* InfV *of its symmetry if and only if* $\mathrm{Zero}(\Delta(\theta; X, \partial U)) \subseteq \mathrm{Zero}(\mathrm{Pr}\mathcal{X}_\theta(\Delta; X, \partial U))$*, i.e., the identity*

$$\mathrm{Pr}\mathcal{X}_\theta(\Delta(\theta; X, \partial U))\big|_{\Delta(\theta;\mathrm{X},\partial\mathrm{U})=0} = 0, \tag{3}$$

holds.

By the standard Lie procedure, we obtain from (3) DTEs of the InfV $\mathcal{X}_\theta$ and denote these by

$$\mathrm{DPS}(\theta) = 0. \tag{4}$$

Consequently, the question of symmetry classification is converted to the one of solving the parametric PDEs (4).

It is known that the solvability of (4) depends on parameter θ. The equations, satisfied by parameter θ and obtained from (4), are called the classifying equations of the symmetry-classification problem of the class of PDEs (2). It is obvious that the specializations $\widetilde{\theta}$ of parameter θ are the solutions to these classifying equations. Hence, a key step toward successfully obtaining the complete symmetry classification is to find all such classifying equations (see examples in Sect. 4).

Summarizing the above discussion, the approach to obtain the complete symmetry classification of a parametric family of PDEs (2) should include the following key steps:

(a) Finding the kernel algebra $\mathcal{X}_0$;
(b) Finding all classifying equations satisfied by the parameters θ;
(c) Finding all extensions $\mathcal{X}_\theta$ of the kernel algebra $\mathcal{X}_0$ according to solutions of the classifying equations.

The result of the application of the above approach is a list of classifying equations (or representatives of their solutions) with corresponding algebras.

We show that the first step (a) is the usual one for solving over-determined systems without arbitrary parameters in determining symmetries of a PDE. Hence existing algorithms and CAS packages can be used to accomplish this step. The second step (b) and third step (c) are our main points of the paper. We use the dchar-set algorithm to investigate them.

3 Dchar-set algorithm for the complete symmetry classification

In this section, we first recall basic results on the dchar-set algorithm of a dps, and then present our algorithm for the complete symmetry classification of a parametric family of PDEs (2). All concepts on dps are taken from [35], [36, Chap. 3], [37].

3.1 Basic results of the dchar-set algorithm (Wu's method)

Under a d-pol rank (or order) $\prec$, a d-pol $f \in \mathcal{K}_X[\partial U]$ is written in the standard form

$$f = I_\alpha (u^\alpha_{k_0})^n + \cdots + I_0,$$

in Wu's theoretical scheme. Here $u^\alpha_{k_0}$ is the highest derivative term in f and is called leading derivative of f. I_α and $\partial f/\partial u^\alpha_{k_0}$ are called the initial and the separant of f, respectively, and $n \in Z_+$ is called the leading power of f. We use IS or IS(DPS) to denote the product of initials and separants (ISP) of a dps DPS ignoring concrete expressions. We use IP to denote the integrability polynomials (conditions) obtained from any two d-pols.

Definition 2 We call d-pol f reduced w.r.t. d-pol g if f does not contain the derivative of the leading derivative of g and the power of the leading derivative of g in f is less than the leading power of g.

Definition 3 A finite dps

$$\text{DCS}: A_1, A_2, \ldots, A_s, \tag{5}$$

is called a differential chain (d-chain) if it satisfies the following two conditions:

(a) $A_1 \prec A_2 \prec \cdots A_s$, and
(b) A_j is reduced w.r.t A_i for $i = 1, 2, \ldots, j-1$.

A d-pol rank induces a rank on a set of d-chains, called a d-chain rank. We still use $\prec$ to denote it [37].

Definition 4 A lowest rank d-chain contained in a dps is called a basic set of the dps.

Suppose dp is a d-pol and DCS is a d-chain. Then Wu's elimination algorithm yields pseudo-reduction formulas [35,37]. That is, there exist d-pols $Q_{\alpha,i} \in \mathcal{K}_X[\partial U]$ such that

$$\text{IS} \cdot \text{dp} = \sum_{i,\alpha,dq_{\alpha,i} \in \text{DCS}} Q_{\alpha,i} D^{\alpha} dq_{\alpha,i} + r, \tag{6}$$

where d-pol r is reduced w.r.t DCS and is called pseudo-remainder of dp w.r.t d-chain DCS, denoted by Prem(dp/DCS), i.e., r = Prem(dp/DCS). If DCS $= \emptyset$, we understand Prem(dp/DCS) = dp. For a dps DPS, we use the notation Prem(DPS/DCS), i.e.,

$$\text{Prem(DPS/DCS)} = \{\text{Prem(dp/DCS) for dp} \in \text{DPS}\}.$$

The following is the definition of a dchar-set of a dps in Wu's method.

Definition 5 If for a dps DPS there exists a d-chain DCS verifying the properties (a_1), (a_2) and (a_3) below:

(a_1) Zero(DPS) $\subset$ Zero(DCS),

(a_2) Prem(DPS/DCS) = 0,

(a_3) Prem(IP/DCS) = 0 for all IP of DCS,

then the d-chain DCS is called a dchar-set of the DPS.

The characteristic set has many algebraic properties [22,37], such as having a triangular structure and containing integrability conditions ..., which make the analysis of the zero set of a dps to be more convenient.

Now we list the basic results used in this article on the dchar-set theory (given by Wu first in [37] and refined in [39–41]).

Theorem 3 [37] *There is an algorithm which permits one to determine a dchar-set* DCS *for a given finite dps* DPS *in a finite number of steps for which the well ordering principle*

$$\begin{aligned} &\text{Zero(DCS/IS)} \subset \text{Zero(DPS)} \subset \text{Zero(DCS)}, \\ &\text{Zero(DPS)} = \text{Zero(DCS/IS)} \cup \text{Zero(DPS, IS)}, \end{aligned} \tag{7}$$

and zero decomposition

$$\text{Zero(DPS)} = \cup_{\text{k}} \text{Zero}(\text{DCS}_{\text{k}}/\text{IS}_{\text{k}}), \tag{8}$$

hold true, where DCS_k *are the dchar-sets of an extension dps obtained by adding some d-pols in* DPS. IS *and* IS_k *are* ISPs *of these dchar-sets, respectively.*

The algorithm mentioned in the above theorem for determining a dchar-set is called the dchar-set algorithm (also called the differential form of Wu's method). It is given through the following algorithmic scheme (W).

Algorithm A: Wu's algorithm for determining a dchar-set of a dps DPS.

$$\begin{array}{cccccccccc} & \text{Step}_0 & & \text{Step}_1 & & \cdots & & \text{Step}_s & & \\ \text{DPS} = & \text{DPS}_0 & \subset & \text{DPS}_1 & \subset & \cdots & \subset & \text{DPS}_s & & \\ & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ & \text{DBS}_0 & \succ & \text{DBS}_1 & \succ & \cdots & \succ & \text{DBS}_s & = \text{DCS} & \\ & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ & \text{RIS}_0 \uparrow & & \text{RIS}_1 \uparrow & & \cdots \uparrow & & \text{RIS}_s = \emptyset, & & \end{array} \qquad (W)$$

in which

$$\text{Step}_i : \begin{cases} \text{DBS}_i \text{ a is base set of } \text{DPS}_i \text{ and } \text{DBS}_{i-1} \succ \text{DBS}_i, \\ R_i = \text{Prem}((\text{DPS}_i \backslash \text{DBS}_i)/\text{DBS}_i) \backslash \{0\}, \\ \text{IT}_i = \text{Prem}(\text{IP}/\text{DBS}_i) \backslash \{0\}, \text{ for any IP of } \text{DBS}_i, \\ \text{RIS}_i = \text{IT}_i \cup R_i, \\ \text{DPS}_i = \text{DPS}_0 \cup \text{DBS}_{i-1} \cup \text{RIS}_{i-1}, i = 0, 1, 2, \ldots, s \end{cases}$$

where $i = 1, 2, \ldots, s$ and $\text{DBS}_{-1} = \text{RIS}_{-1} = \emptyset$; s represents the number of calculation steps. The down-arrow means that computation is continuous in this step and the up-arrow shows the computation turn into next loop step. The algorithm is implemented in [42, Chap. 42] as part of CAS MMP.

The rank of a d-pol has been playing a key role in Wu's method mentioned above. A natural choice of such a rank is the diff-graded lex order which is induced by the total derivative differential form graded lexicographic rank on derivative terms [22, Chap. 5], [37], [38, Chap. 2], [40]. In all examples in this paper, we take the rank as the d-pol rank. The decomposition (8) and Algorithm A are the main tools for solving symmetry-classification problems in this paper.

In the following, we give illustrative examples to show the efficiency of Theorem 3 and Algorithm A in determining the zero set of an over-determined system and also present the well ordering (triangular) structure of a dchar-set.

Example 1 Consider a dps

$$\text{DPS} = \left\{ \begin{array}{l} \xi_v - \tau_u, \eta_u - \phi_v + \xi_x - \tau_t, \eta_v + u(\eta_t - \phi_x) + \tau_x, u^2\xi_u - \tau_v, u^2\phi_u - u\tau_u - \eta_v, \\ u\xi_v + u^2\xi_t - \tau_x, u(\eta_u - \phi_v - \xi_x + \tau_t) + 2(\tau_v - \eta), u(\phi_v - \tau_t) - (\tau_v + \eta_x - \eta) + u^2\phi_t. \end{array} \right\}$$

in $\mathcal{K}_X[\partial U]$ with $X = (x, t, u)$ and $U = (\xi, \phi, \eta, \tau)$. Under the basic rank $x \prec t \prec u \prec \xi \prec \phi \prec \eta \prec \tau$, executing Algorithm A, we obtain a dchar-set of DPS as follows:

$$\text{DCS} = \left\{ \begin{array}{lll} \xi_{tv}, \xi_{tt}, \xi_{xt}, & \phi_v, \phi_t, \phi_u + 2\xi_t, & \\ \xi_t + u\xi_{tu}, & \phi_x + 2u\xi_t; & \tau_x - u\xi_v - u^2\xi_t, \\ \xi_v + u\xi_{uv} + u\xi_t - \xi_{xv}, & & \tau_t + u\xi_u - \xi_x - u^{-1}\eta, \\ \xi_{vv} + \xi_x - \xi_{xx}, & \eta_x + u\xi_x, \eta_t + u\xi_t, & \tau_u - \xi_v, \tau_v - u^2\xi_u, \\ \xi_x + u^2\xi_{uu} + 2u\xi_u - \xi_{xx}, & u\eta_u - \phi + u^2\xi_u, & \\ \xi_x + u\xi_{xu} - \xi_{xx}; & \eta_v + u\xi_v + 2u^2\xi_t; & \end{array} \right\}$$

with $\text{IS} = u$. Hence one has

$$\text{Zero(DPS)} = \text{Zero(DCS)},$$

by the well-ordering principle in Theorem 3. This shows the equivalence between solving $\text{DPS} = 0$ and $\text{DCS} = 0$. The well-ordering (triangular form) structure of the dchar-set DCS is seen from its four parts. The first part consists of the first eight d-pols involving ξ only. The second part is the next four d-pols only involving ξ and ϕ. The third part follows from the four d-pols involving ξ, ϕ and η. The fourth part is the last four d-pols involving ξ, ϕ, η and τ. Obviously, the equivalence and well-ordered (triangular) structure of the dchar-set DCS make the determination of Zero(DPS) easier through solving Zero(DCS). The zero set of ξ is obtained from the first part of DCS; the zero sets of ϕ, η, τ are obtained from the following parts of the DCS sequentially by using the previously determined zero sets step by step.

Example 2 A proper example for illustrating the adaptability of the algorithm to a nonlinear case is determining the nonclassical symmetries of Burgers equation $\Delta = u_t + uu_x + u_{xx} = 0$. This problem was considered in [16, pp. 320–321], [43]. Here we give its solution again through use of the dchar-set algorithm, which provides a different point of view for the same problem. Let $\mathcal{X} = \partial_t + \xi(x,t,u)\partial_x + \eta(x,t,u)\partial_u$ be InfV for nonclassical symmetries of the Burgers equation with the invariant surface condition $u_t - \eta(x,t,u) + \xi(x,t,u)u_x = 0$. By the standard procedure given in (3), we have DTEs DPS=0 for $\mathcal{X}$. Here

$$\mathrm{DPS} = \{\xi_{uu}, \eta_t + u\eta_x + \eta_{xx} + 2\eta\xi_x, \eta_{uu} + 2u\xi_u - 2\xi\xi_u - 2\xi_{xu}, 2\eta_{xu} + 2\eta\xi_u - \xi_t + u\xi_x - 2\xi\xi_x - \xi_{xx} + \eta\}.$$

Executing the Algorithm A, under the rank $x \prec t \prec u \prec \xi \prec \eta$, we get the decomposition

$$\mathrm{Zero(DPS)} = \mathrm{Zero(DCS_1)} \cup \mathrm{Zero(DCS_2)} \cup \mathrm{Zero(DCS_3)}.$$

The three different characteristic sets $\mathrm{DCS_1}$, $\mathrm{DCS_2}$ and $\mathrm{DCS_3}$ of the DPS are given by

$$\mathrm{DCS_1} = \{\eta, u - \xi\},$$
$$\mathrm{DCS_2} = \{\xi_u, \xi_{xx}, \eta_{xx}, \eta_u + \xi_x, \eta_t + u\eta_x + 2\eta\xi_x, \eta - \xi_t + u\xi_x - 2\xi\xi_x\},$$
$$\mathrm{DCS_3} = \{\eta_{uu} - u + \xi, 1 + 2\xi_u, \eta_t + u\eta_x + \eta_{xx} + 2\eta\xi_x, 2\eta_{xu} - \xi_t + u\xi_x - 2\xi\xi_x - \xi_{xx},$$
$$2\eta_{tu} + 2\eta_x + u\xi_t + 4\eta_u\xi_x - u^2\xi_x + 2u\xi\xi_x + 2\xi_x^2 + \xi_{xt} + 2\xi\xi_{xx} + \xi_{xxx}\}.$$

Solving the equations of the well-ordered system $\mathrm{DCS_1} = 0$, $\mathrm{DCS_2} = 0$, $\mathrm{DCS_3} = 0$, respectively, we obtain the infinitesimal functions ξ and η as follows:

$$\mathrm{Zero(DCS_1)} = \{\xi = u, \eta = 0\},$$
$$\mathrm{Zero(DCS_2)} = \left\{\xi = \frac{c_1tx + c_2x + c_4t + c_5}{c_1t^2 + 2c_2t + c_3}; \eta = \frac{c_1(x - tu) - c_2u + c_4}{c_1t^2 + 2c_2t + c_3}\right\},$$
$$\mathrm{Zero(DCS_3)} = \left\{\xi = -\frac{1}{2}u + \alpha(x,t), \eta = \frac{1}{4}u^3 - \frac{1}{2}\alpha(x,t)u^2 - \beta(x,t)u + \gamma(x,t)\right\},$$

where $\alpha(x,t)$, $\beta(x,t)$, $\gamma(x,t)$ satisfy the PDE system

$$\begin{cases} \alpha_t(x,t) + \alpha_{xx}(x,t) + 2\beta_x(x,t) + 2\alpha(x,t)\alpha_x(x,t) = 0, \\ \beta_t(x,t) + \beta_{xx}(x,t) - \gamma_x(x,t) + 2\beta(x,t)\alpha_x(x,t) = 0, \\ \gamma_t(x,t) + \gamma_{xx}(x,t) + 2\gamma(x,t)\alpha_x(x,t) = 0. \end{cases} \tag{9}$$

It is interesting that the zero point set $\mathrm{Zero(DCS_2)}$ corresponds to all finite-dimensional classical symmetries of the Burgers equation. All nonclassical symmetries of the equation arise from the dchar-sets $\mathrm{DCS_1}$ and $\mathrm{DCS_3}$. These three parts correspond to the cases given in [16, pp. 320–321], [43] obtained in a different way. Particularly, by taking 1). $\alpha(x,t) = a_0t^2 + a_1t + a_2$; 2). $\alpha(x,t) = \beta(x,t) = \gamma(x,t) = 0$; 3). $\alpha(x,t) = 0, \beta(x,t) = a_3$; 4). $\alpha(x,t) = a_4, \beta(x,t) = \gamma(x,t) = 0$; 5). $\alpha(x,t) = 1/x, \beta(x,t) = \gamma(x,t) = 0$; 6). $\beta(x,t) = \gamma(x,t) = 0$ in (9) (here $a_i, i = 0,1,2,3,4$ are arbitrary constants), we get the nonclassical symmetries reported in [16, pp. 320–321]. Solving for different solutions to (9), one gets new nonclassical symmetries of the equation. For example, by taking $\alpha(x,t) = xt^{-1}$ in (9), we have new nonclassical symmetries with $\beta(x,t) = (-x^2/4 + c_2t + c_3)t^{-2}$, $\gamma(x,t) = (c_1 + c_2x - x/2)t^{-2}$ in $\mathrm{Zero(DCS_3)}$. Here $c_i, i = 1,2,3$ are arbitrary constants.

3.2 Dchar-set algorithm for the complete symmetry classification

In the following, based on Theorem 3 and the conversion of symmetry classification to determining the zero set $\mathrm{Zero(DPS(\theta))}$ (see (4)), we give our algorithm for the complete symmetry classification of the family of PDEs (2) through decomposing $\mathrm{Zero(DPS(\theta))}$ into a union of a series of zero sets of dchar-sets of the parametric dps $\mathrm{DPS}(\theta)$. The algorithm mainly consists of three steps.

Step 1 Producing DTEs $\mathrm{DPS}(\theta) = 0$. This is a routine task for Lie's standard algorithm [1, Sect. 2.4], [2, Chap. 4].

Step 2 Determining the kernel algebra $\mathcal{X}_0$. Regard $\mathrm{DPS}(\theta) = 0$ as an identity in θ (θ is arbitrary), and let the coefficients of θ and its derivatives be zero. Doing this, we get an extended system of PDEs and denote it as $\mathrm{DCS}_0 = 0$. Obviously, for arbitrary θ we have

$$\mathrm{Zero}(\mathrm{DCS}_0) \subseteq \mathrm{Zero}(\mathrm{DPS}(\theta)). \tag{10}$$

Hence, $\mathrm{DCS}_0 = 0$ corresponds to the kernel G_0. Solving it (without parameters), we find the kernel algebra $\mathcal{X}_0$.

Step 3 Determining the extensions InfV $\mathcal{X}_\theta$ of $\mathcal{X}_0$ for the specializations of the parameter θ.

To do this, we decompose the zero set $\mathrm{Zero}(\mathrm{DPS}(\theta))$ in terms of zero sets of dchar-sets as in (8) by using a proper improvement of algorithm A. The improvement comes from the simplifications of computation for overcoming the complexity of symbolic computation. In procedure (W), as discovered by Collins [44,45], the remainders R_i and IT_i have leading coefficients of their proceeding d-pols in DBS_i as factors. From the inevitable occurrence of such factors, the d-pols in the procedure become too large which render the computation difficult to be carried on further. Thus, it is worthy to remove all such factors during the procedure to render smaller d-pols. Therefore, we have to pay special attention to the so-called 'removed factors' occurring in implementation of the scheme (W). This means, in the execution of Algorithm A on $\mathrm{DPS}(\theta)$, we may encounter the factorable d-pol $\mathrm{dp} \in \mathrm{DPS}_i(\theta)$ for some i, such as,

$$\mathrm{dp} = Q(\theta) \cdot P,$$

where $P \in \mathcal{K}_X[\partial U]$ and $Q(\theta)$, called a removed factor, comes from IS terms of d-pols in the basic set DBS_i and depends on parameter θ. We remove the factor $Q(\theta)$ by means of replacing dp with P, i.e., simply by resetting $\mathrm{dp} = P$ in $\mathrm{DPS}_i(\theta)$. Then we continue the algorithm on the simplified dps $\mathrm{DPS}_i(\theta)$ until ending the procedure of Algorithm A. In the whole procedure we always do such a removing as long as it occurs. Here we denote the product of all such removed factors through the notation $\mathrm{RF}(\theta)$. As a result, according to the well-ordering principle (7) in Theorem 2, we have the decomposition

$$\mathrm{Zero}(\mathrm{DPS}(\theta)) = \mathrm{Zero}(\mathrm{DCS}_1(\theta)/\mathrm{IS}_1(\theta) * \mathrm{RF}(\theta)) \cup \mathrm{Zero}(\mathrm{DPS}(\theta), \mathrm{IS}_1(\theta)) \cup \mathrm{Zero}(\mathrm{DPS}(\theta), \mathrm{RF}(\theta)),$$

after the first execution of Algorithm A. Here $\mathrm{IS}_1(\theta)$ is the ISP of $\mathrm{DCS}_1(\theta)$. After removing the non-zero factors in $\mathrm{IS}_1(\theta) * \mathrm{RF}(\theta)$, we rewrite the above decomposition as

$$\mathrm{Zero}(\mathrm{DPS}(\theta)) = \mathrm{Zero}(\mathrm{DCS}_1(\theta)/\Pi_j I_{1j} * \Pi_k I_k^1) \cup_j \mathrm{Zero}(\mathrm{DPS}(\theta), I_{1j}) \cup_k \mathrm{Zero}(\mathrm{DPS}(\theta), I_k^1), \tag{11}$$

where $I_{1j} \in \mathcal{K}_X[\partial U]$ and I_k^1 depends on parameter θ and its derivatives only. If $\mathrm{DCS}_1(\theta)$ corresponds to the kernel G_0, then from (10) we delete the first part of the above decomposition. This cancelation is always done as long as the case of dchar-set corresponds to the kernel algebra in further decompositions.

For the non-dchar-sets (second and third parts) of the above decomposition, we use Algorithm A again to obtain a further decomposition. For the second part we compute the dchar-set of the extended dps $\mathrm{DPS}(\theta) \cup I_{1j}$. In the third part, we compute the dchar-set of $\mathrm{DPS}(\theta)$ under the side condition $I_k^1 = 0$ (i.e., the classifying equation). We obtain a further decomposition of (11) in terms of zero sets of dchar-sets of $\mathrm{DPS}(\theta)$ and zero sets of some non-dchar-sets. On these non-dchar-set parts in a subsequent decomposition, we repeat the same procedure further and further. Wu's results (8) and algorithm scheme (W) guarantee that the above procedure terminates in finitely many steps. Hence we obtain the final decomposition

$$\mathrm{Zero}(\mathrm{DPS}(\theta)) = \cup_i \mathrm{Zero}(\mathrm{DCS}_i(\theta)/\Pi_j I_{ij}) \cup_k \mathrm{Zero}(\{\mathrm{DCS}'_k(\theta), \mathrm{ClE}_k(\theta)\}/\Pi_l I_l^k), \tag{12}$$

where $\mathrm{DCS}_i(\theta)$ is the dchar-set of $\mathrm{DPS}(\theta)$ without associated classifying equations; $\mathrm{DCS}'_k(\theta)$ is the dchar-set of $\mathrm{DPS}(\theta)$ with the associated classifying equations $\mathrm{ClE}_k(\theta) = 0$; I_{ij} and I_l^k are d-pols obtained from the ISP of $\mathrm{DCS}_i(\theta)$, $\mathrm{DCS}'_k(\theta)$ and removed factors $\mathrm{RF}(\theta)$. Thus, we have the complete decomposition of the solution set of $\mathrm{DPS}(\theta) = 0$ in terms of zero sets of dchar-sets through expression (12) and all classifying equations $\mathrm{ClE}_k(\theta) = 0$ appearing in the parts of $\mathrm{DCS}'_k(\theta)$. This yields the complete symmetry classification of the family of PDEs in terms of the dchar-sets decomposition (12). Each branch of the right-hand sides of (12) corresponds to a class of symmetries and associated parameters.

The above procedure yields the following theorem and algorithm:

Theorem 4 (Classification theorem) *For a given family of PDEs* (2) *with parameter* θ, *let* $\mathrm{DPS}(\theta) = 0$ *be the set of DTEs of* $\mathrm{Inf}V\ \mathcal{X}_\theta$ *for the maximal symmetry* G_θ. *Then the decomposition* (12) *yields the complete symmetry classification of the family of PDEs in the sense that each class of symmetries corresponds to a dchar-set.*

Algorithm B: The complete symmetry classification for a parametric family of PDEs.
Input: A family of PDEs $\Delta(\theta; X, \partial U) = 0$ with some parameter θ.
Output: Classifying equations and corresponding classification branches.
Begin
Step 1: Produce DTEs $\mathrm{DPS}(\theta) = 0$ (Use the standard Lie's algorithm in (3));
Step 2: Determine the kernel algebra $\mathcal{X}_0$ (Use Algorithm A to solve $\mathrm{DCS}_0 = 0$);
Step 3: Construct the decomposition (12) for the dps $\mathrm{DPS}(\theta)$ (Use Algorithms A);
End

This algorithm gives a mechanical way to directly and systematically obtain the complete symmetry classification for a given parametric family of PDEs in the sense of obtaining all classifying equations and corresponding well-ordered equations of $\mathrm{Inf}V\ \mathcal{X}_\theta$.

In order to obtain final specific expressions of $\mathrm{Inf}V\ \mathcal{X}_\theta$, one has to explicitly determine each branch in decomposition (12). This is equivalent to solving $\mathrm{DCS}_i(\theta) = 0$ and $\mathrm{DCS}'_k(\theta) = 0$ with $\mathrm{ClE}_k(\theta) = 0$. Because of the well-ordered structure (see Examples 1, 2) of the dchar-sets, these PDEs are more easily solved than the original system $\mathrm{DPS}(\theta) = 0$. Although this integration procedure is not constructive in general [23], the well-ordered structure provides us with more help to get explicit solutions to these equations and enhance the efficiency in the applications of existing algorithms and CAS packages. In our examples (see Sect. 4), all explicit integrations of equations corresponding to dchar-sets are obtained by hand calculation. Actually, one may use many techniques to simplify integrations. One of them is to use equivalence transformations. If the solutions to each classifying equation can be solved in advance, then we use a representative of the solutions under admitted equivalence transformations to determine the corresponding symmetries. This significantly simplifies the classifying computation (see example in Sect. 4.2).

4 Applications

We give some illustrative examples to show applications and the efficiency of Algorithm B. In the rest of this paper, we use c_i to denote an arbitrary constant in a symmetry.

4.1 Potential symmetry classification of a linear wave equation with a parameter

We give a potential symmetry classification of the wave equation

$$u_{xx} = H(x)u_{tt}, \tag{13}$$

with parameter $\theta = H(x) \not\equiv 0$ to illustrate our classification algorithm. An equivalent potential system of the equation is

$$v_t - u_x = 0,\ v_x - H(x)u_t = 0. \tag{14}$$

Suppose that PDE system (14) admits a classical symmetry with InfV

$$\mathcal{X} = \xi(x,t,u,v)\frac{\partial}{\partial x} + \tau(x,t,u,v)\frac{\partial}{\partial t} + \eta(x,t,u,v)\frac{\partial}{\partial u} + \phi(x,t,u,v)\frac{\partial}{\partial v}. \tag{15}$$

Now, our task is to determine functions ξ, τ, η, ϕ in (15) and H in (14) and show whether or not the original equation (13) admits potential symmetries, i.e., at least one of the functions ξ, τ, η depends on the potential variable v.

4.1.1 Step 1: Producing DTEs

From (3), we get the determining equations DPS = 0. Here

$$\mathrm{DPS} = \left\{ \begin{array}{l} \xi_u - \tau_v,\, H(x)\xi_v - \tau_u,\, \phi_t - \eta_x, \\ \xi_x - \tau_t - \eta_u + \phi_v,\, H(x)(\xi_t - \eta_v) - \tau_x + \phi_u, \\ H(x)(\xi_t + \eta_v) - \tau_x - \phi_u,\, \phi_x - H(x)\eta_t, \\ H(x)(\xi_x - \tau_t + \eta_u - \phi_v) + H'(x)\xi, \end{array} \right\} \tag{16}$$

which depends on parameter $\theta = H(x)$.

4.1.2 Step 2: Determining the kernel algebra $\mathcal{X}_0$

For arbitrary parameter $\theta = H(x)$, the DTEs DPS = 0 are further reduced to $\mathrm{DCS}_0 = 0$. Here

$\mathrm{DCS}_0 = \{\xi, \tau_x, \tau_t, \tau_u, \tau_v, \eta_v, \eta_t, \phi_u, \phi_x, \eta_u - \phi_v, \phi_t - \eta_x\}$.

Hence, by solving $\mathrm{DCS}_0 = 0$, the infinitesimal functions of the principal symmetry are easily obtained in terms of the zero set,

$$\mathrm{Zero}(\mathrm{DCS}_0) = \{\xi = 0, \tau = c_1, \eta = c_2 x + c_3 u + c_4, \phi = c_2 t + c_3 v + c_5\}\,. \tag{17}$$

4.1.3 Step 3:Determining extensions of the kernel algebra $\mathcal{X}_0$

4.1.3.1 Step 3.1: Calculating the zero decomposition of the dps DPS. We execute the Algorithm B under the rank $x \prec t \prec \xi \prec \tau \prec \phi \prec \eta$, and obtain the zero decomposition (12) given by

$$\mathrm{Zero}(\mathrm{DPS}) = \mathrm{Zero}(\mathrm{DCS}_1/I_{11}) \cup \mathrm{Zero}(\{\mathrm{DCS}_2, I_{11}\}/I_{21}) \cup \mathrm{Zero}(\mathrm{DCS}_3, I_{21}), \tag{18}$$

where

$$\mathrm{DCS}_1 = \left\{ \begin{array}{l} \xi, \tau_x, \tau_t, \tau_u, \tau_v, \eta_v, \phi_u, \phi_{vv}, \phi_{tv}, \phi_{xv}, \\ \eta_u - \phi_v, \eta_x - \phi_t, H(x)\eta_t - \phi_x, \\ H(x)(H(x)\phi_{tt} - \phi_{xx}) + H'(x)\phi_x, \end{array} \right\},$$

$$\mathrm{DCS}_2 = \left\{ \begin{array}{l} \tau_u, \tau_v, \tau_x + 2\phi_u, 2H(x)H'(x)\tau_t - (3H'(x)^2 - 2H(x)H''(x))\xi, \\ \eta_x - \phi_t, H(x)\eta_t - \phi_x, 2H(x)(\eta_u - \phi_v) + H'(x)\xi, H(x)\eta_v - \phi_u, \\ \xi_u, \xi_v, H(x)\xi_t + 2\phi_u, H(x)H'(x)\xi_x - (H'(x)^2 - H(x)H''(x))\xi, \\ \phi_{vv}, \phi_{uu}, \phi_{uv}, H(x)(H(x)\phi_{tt} - \phi_{xx}) + H'(x)\phi_x, \\ H(x)H'(x)\phi_{xu} - (2H'(x)^2 - H(x)H''(x))\phi_u, \\ H(x)^2H'(x)\phi_{tv} - (H'(x)^2 - H(x)H''(x))\phi_u, \\ 2H(x)H'(x)^2\phi_{tu} - (H'(x)^2H''(x) - 2H(x)H''(x)^2 + H(x)H'(x)H^{(3)}(x))\xi, \\ 2H(x)H'(x)^2\phi_{xv} - (H'(x)^2H''(x) - 2H(x)H''(x)^2 + H(x)H'(x)H^{(3)}(x))\xi. \end{array} \right\},$$

$$\mathrm{DCS}_3 = \left\{ \begin{array}{l} \xi_{tu} - \xi_{xv}, H(x)\xi_{vv} - \xi_{uu}, H(x)\xi_{tv} - \xi_{xu}, H(x)\xi_{tt} - \xi_{xx}, \\ H(x)\xi_t - \tau_x, \xi_x - \tau_t, H(x)\xi_v - \tau_u, \xi_u - \tau_v, \\ \eta_x - \phi_t, H(x)\eta_t - \phi_x, \eta_u - \phi_v, H(x)\eta_v - \phi_u, \\ \phi_{tu} - \phi_{xv}, H(x)\phi_{vv} - \phi_{uu}, H(x)\phi_{tv} - \phi_{xu}, H(x)\phi_{tt} - \phi_{xx}, \end{array} \right\},$$

with

$$\begin{aligned} I_{11} = {} & 2H'(x)^4H''(x) - 2H(x)H'(x)^2H''(x)^2 - 4H(x)^2H''(x)^3 + 5H(x)^2H'(x)H''(x)H^{(3)}(x) \\ & - H(x)^2H'(x)^2H^{(4)}(x); \end{aligned}$$

$$I_{21} = H'(x).$$

4.1.3.2 Step 3.2: Solving for the zero sets in the zero decomposition. It is noticed that in the first branch $\mathrm{Zero}(\mathrm{DCS}_1/I_{11})$ in (18) has no classifying equation. In the second branch $\mathrm{Zero}(\mathrm{DCS}_2, I_{11}/I_{21})$, there is a classifying equation $\mathrm{ClE}_1(\theta) = I_{11} = 0$. In the third branch Zero(DCS_3, I_{21}), there is a classifying equation $\mathrm{ClE}_2(\theta) = I_{21} = 0$. The set of equations $\mathrm{DCS}_1 = 0$; $\mathrm{DCS}_2 = 0$; $\mathrm{DCS}_3 = 0$ is a well-ordered system which is more easily integrated by hand, viz.,

$$\mathrm{Zero}(\mathrm{DCS}_1/I_{11}) = \{\xi = 0, \tau = c_1, \eta = c_2 u + a(x,t), \phi = c_2 v + b(x,t)\}, \tag{19}$$

where $(a(x,t), b(x,t))$ is any solution of (14);

$$\mathrm{Zero}(\{\mathrm{DCS}_2, I_{11}\}/I_{21}) = \left\{ \xi = g(t)\frac{H(x)}{H'(x)}, \tau = \frac{3H'(x)^2 - 2H(x)H''}{2H'(x)^2}\int^t g(s)\mathrm{d}s + c_1, \right.$$
$$\eta = \left(\frac{H(x)H'' - 3H'(x)^2}{2H'(x)^2}g(t) + c_2\right)u - \frac{H(x)}{2H'(x)}g'(t)v + a(x,t),$$
$$\left. \phi = -\frac{H(x)^2}{2H'(x)}g'(t)u + \left(\frac{H(x)H'' - H'(x)^2}{2H'(x)^2}g(t) + c_2\right)v + b(x,t)\right\}, \tag{20}$$

where σ is an arbitrary constant and $g(t)$ satisfies

$$\frac{H'(x)}{H(x)^2}\left(\frac{3H'(x)^2 - 2H(x)H''(x)}{2H'(x)^2}\right)' = \sigma = \frac{g''(t)}{g(t)}. \tag{21}$$

Here the parameter $H(x)$ satisfies $I_{11} = 0$ and (21). Since (21) implies $I_{11} = 0$, it follows that (21) is equivalent to the equation $I_{11} = 0$;

For $I_{21} = 0$, let $H(x) = \alpha^2$. Then

$$\mathrm{Zero}(\mathrm{DCS}_3, I_{21}) = \left\{ \begin{array}{l} \xi = f(x,t,\alpha u - v) + g(x,t,\alpha u + v), \\ \tau = \alpha(-f(x,t,\alpha u - v) + g(x,t,\alpha u + v)) + F(x,t), \\ \eta = \frac{1}{\alpha}(-\bar{f}(x,t,\alpha u - v) + \bar{g}(x,t,\alpha u + v)) + \bar{F}(x,t), \\ \phi = \bar{f}(x,t,\alpha u - v) + \bar{g}(x,t,\alpha u + v), \end{array} \right\}, \tag{22}$$

where functions $(y,z) = (f,g)$ and $(y,z) = (\bar{f},\bar{g})$ satisfy

$$y_x + \alpha y_t = \frac{1}{2}(k(x,t) + h(x,t)), \quad z_x - \alpha z_t = \frac{1}{2}(k(x,t) - h(x,t)), \tag{23}$$

and $P = F(x,t)$ and $P = \bar{F}(x,t)$ are any two solutions of the system

$$P_x(x,t) = \alpha h(x,t), P_t(x,t) = k(x,t), \tag{24}$$

in which $k(x,t)$ and $h(x,t)$ are arbitrary functions.

Now, we get the complete potential symmetry classification of the wave equation (13) through (14), (18), and (19), (20) with (21), (22) with (23) and (24). The classification results are summarized in the following Table 1. Note that the dchar-sets DCS_0 and DCS_1 do not yield potential symmetries. If $g'(t) \neq 0$, then the dchar-set DCS_2 yields potential symmetries for equation (13). The dchar-set DCS_3 yields an infinite number of potential symmetries.

These results cover those given in [2, pp. 182–188]. It is noticed that the symmetries corresponding to DCS_3 obtained in our classification were not mentioned in [2, pp. 182–188]. In addition, it is worthy to note that the solutions in (19), (20) and (22) present unifying expressions for the symmetries corresponding to various parameters $H(x)$ of the solutions of (14), (21), (23) and (24), respectively.

Table 1 Table potential symmetry classification of (13)

$H(x)$	dchar-set	ξ, τ, η, ϕ	Condition	Potential
Arbitrary	DCS_0	(17)	No	No
$I_{11} \neq 0$	DCS_1	(19)	$a, b \in$ Zero (14)	No
$I_{11} = 0, I_{21} \neq 0$ ⇔ (21)	DCS_2	(20)	$\sigma, g \in$ Zero (21), $a, b \in$ Zero (14)	Yes, $g' \neq 0$
$I_{21} = 0$	DCS_3	(22)	$(f,g), (\bar{f},\bar{g}) \in$ Zero (23), $(F,\bar{F}) \in$ Zero (24)	Yes, infinite

4.2 Potential symmetry classification of a nonlinear wave equation with a parameter

We consider a potential symmetry classification of the nonlinear wave equation $u_{tt}+u_t=(F(u)u_x)_x$. The classical symmetry classification of this equation was considered in [46]. A potential system is given by

$$v_t=F(u)u_x,\quad v_x=u_t+u. \tag{25}$$

4.2.1 Step 1: Producing DTEs

By the standard Lie algorithm, we get the DTEs DPS=0 for the classical symmetries with InfV $X=\xi(x,t,u,v)\partial_x+\tau(x,t,u,v)\partial_t+\eta(x,t,u,v)\partial_u+\phi(x,t,u,v)\partial_v$ for the potential system (25). Here

$$\mathrm{DPS}=\left\{\begin{array}{ll} F(u)(\eta_x+u\tau_x)+u\phi_u-\phi_t, & F(u)\tau_v-\xi_u,\\ F(u)(\eta_v+u\tau_v)-\phi_u, & \eta_u-\phi_v+\xi_x+2u\tau_u-\tau_t,\\ F(u)\tau_x+u\xi_u-\xi_t, & u\eta_u-\eta_t+\phi_x+u^2\tau_u-u\tau_t-\eta,\\ F(u)(\eta_u-\phi_v-\xi_x+\tau_t)+F'(u)\eta, & \xi_v-\tau_u. \end{array}\right\}.$$

4.2.2 Step 2: Determining the kernel algebra $\mathcal{X}_0$

Obviously, this algebra corresponds to

$\mathrm{DCS}_0=\{\xi_x,\xi_t,\xi_u,\xi_v,\tau_x,\tau_t,\tau_u,\tau_v,\phi_x,\phi_t,\phi_u,\phi_v,\eta\}=0,$

which yields the infinitesimal functions of $\mathcal{X}_0$ given by

$\mathrm{Zero}(\mathrm{DCS}_0)=\{\xi=c_1,\tau=c_2,\phi=c_3,\eta=0\}.$

4.2.3 Step 3: Determining the extended algebra $\mathcal{X}_\theta$

Under the basic rank $\xi\prec\eta\prec\tau\prec\phi$, we execute Algorithm B on the dps DPS and firstly obtain the decomposition

$$\mathrm{Zero(DPS)=Zero(DPS,\ IS_1)}, \tag{26}$$

in which

$$\mathrm{IS}_1=\mathrm{IS}_2F(u)F^{(4)}(u)-3F(u)^2F^{(3)}(u)^2+2F'(u)(8F(u)F''(u)-3F'(u)^2)F^{(3)}(u)\\ +6F''(u)^2(F'(u)^2-2F(u)F''(u)).$$

Note that all classifying equations come from $\mathrm{IS}_1=0$. Using again the algorithm for the dps under $\mathrm{IS}_1=0$, we have the further decomposition

$$\mathrm{Zero(DPS,IS_1)}=\mathrm{Zero(DCS_1,\ IS_1/IS_3*IS_4)}\cup\mathrm{Zero(DPS,\ IS_3/IS_2)}\cup\mathrm{Zero(DPS,\ IS_4/IS_2)}\\ \cup\mathrm{Zero(DPS,\ IS_2)}. \tag{27}$$

Here the dchar-set $\mathrm{DCS}_1=0$ is given by

$$\mathrm{DCS}_1=\left\{\begin{array}{l} \eta_v,\eta_t,\eta_x,\phi_u,\phi_t,\xi_u,\xi_t,\tau_v,\tau_t,\tau_x,\\ \mathrm{IS}_2F(u)\eta_u+(3F'(u)^3-4F(u)F'(u)F''(u)+F(u)^2F^{(3)}(u))\eta,\\ 2\mathrm{IS}_2F(u)\xi_v+(F'(u)^2F''(u)-2F(u)F''(u)^2+F(u)F'(u)F^{(3)}(u))\eta,\\ 2\mathrm{IS}_2F(u)\xi_u+\eta(F'(u)^2F''(u)-2F(u)F''(u)^2+F(u)F'(u)F^{(3)}(u)),\\ 2\mathrm{IS}_2F(u)\xi_x+[F'(u)(3F'(u)^2-2F(u)F''(u)-uF'(u)F''(u)-uF(u)F^{(3)}(u))\\ \quad+2uF(u)F''(u)^2]\eta,\\ 2\mathrm{IS}_2F(u)\phi_v+\eta(9F'(u)^3-10F(u)F'(u)F''(u)+uF'(u)^2F''(u)-2u\,F(u)F''(u)^2\\ \quad+2F(u)^2F^{(3)}(u)+uF(u)F'(u)F^{(3)}(u)),\\ 2\mathrm{IS}_2F(u)\phi_x+\eta(6F(u)F'(u)^2-6uF'(u)^3-4F(u)^2F''(u)+8uF(u)F'(u)F''(u)\\ \quad-u^2F'(u)^2F''(u)+2u^2F(u)F''(u)^2-2uF(u)^2F^{(3)}(u)-u^2\,F(u)F'(u)F^{(3)}(u)) \end{array}\right\}$$

and

$\mathrm{IS}_2 = 2F(u)F''(u) - 3F'(u)^2,$
$\mathrm{IS}_3 = F'(u)^2\, F''(u) - 2\, F(u)\, F''(u)^2 + F(u)\, F'(u)\, F^{(3)}(u),$
$\mathrm{IS}_4 = 6\, F'(u)^3 - 6\, F(u)\, F'(u)\, F''(u) + F(u)^2\, F^{(3)}(u).$

At this point, the use of equivalence transformations admitted by system (25) can simplify subsequent computations. It is observed that system (25) admits linear equivalence transformations

$$x' = ax + b, t' = t + d, u' = lu + m, v' = alv + amx + p, F' = F/a^2, \tag{28}$$

where a, b, d, l, m, p are arbitrary constants $al \neq 0$. In this example, we show how equivalence transformations involving scalings and translations in u and scalings in F simplify the computations.

Solving the classifying equations $\mathrm{IS}_3 = 0$, $\mathrm{IS}_4 = 0$ and $\mathrm{IS}_2 = 0$, we can select representatives of their solutions under equivalence transformation (28) as $F(u) = u^\alpha$, $F(u) = \mathrm{e}^u$ for $\mathrm{IS}_3 = 0$ in which α is an arbitrary constant; $F(u) = u^{-2}$ and $F(u) = u^{-1}$ for $\mathrm{IS}_4 = 0$; and $F(u) = u^{-2}$ for $\mathrm{IS}_2 = 0$.

The classifying equation $\mathrm{IS}_1 = 0$ can be reduced to

$$2(c_2 + c_1 u)F(u) + (c_1 u^2 + (c_2 - c_3)u - c_4)F'(u) = 0, \tag{29}$$

for arbitrary constants c_i, $i = 1, 2, \ldots, 5$.

For simplicity, for $c_1 \neq 0$, we let $b = (c_2 - c_3)/(2c_1)$, $\Delta = -c_4/c_1 - b^2$. Then the solutions of the classifying equation (29) lead to the following cases.

4.2.4 Case A: $c_1 \neq 0$

4.2.4.1 Subcase A.1: For $\Delta > 0$,

$$F(u) = k\left((u+b)^2 + \Delta\right)^{-1} \exp\left(-\frac{c_2 + c_3}{\sqrt{\Delta}} \arctan \frac{u+b}{\sqrt{\Delta}}\right).$$

Here and in the sequel k is an integration constant. This case is equivalent to $F(u) = (u^2+1)^{-1}\mathrm{e}^{\alpha \arctan u}$ under equivalence transformations (28) for any real number α;

4.2.4.2 Subcase A.2: For $\Delta < 0$,

$$F(u) = k\left((u+b)^2 + \Delta\right)^{-1} \exp\left(\frac{c_2 + c_3}{\sqrt{-\Delta}} \operatorname{arctanh} \frac{u+b}{\sqrt{-\Delta}}\right).$$

This case is equivalent to $F(u) = (1-u^2)^{-1}\mathrm{e}^{\alpha \operatorname{arctanh} u}$ under equivalence transformations (28) for any real number α;

4.2.4.3 Subcase A.3: For $\Delta = 0$,

$$F(u) = k\,(u+b)^{-2} \exp\left(\frac{c_2 + c_3}{u+b}\right).$$

This case is equivalent to $F(u) = u^{-2}\mathrm{e}^{\frac{1}{u}}$ under equivalence transformations (28);

4.2.5 Case B: $c_1 = 0$

4.2.5.1 Subcase B.1: For $c_2 - c_3 \neq 0$,

$$F(u) = k((c_2 - c_3)u - c_4)^{-\frac{2c_2}{c_2 - c_3}}.$$

This is equivalent to $F(u) = u^\alpha$ under equivalence transformations (28) for any real number α;

4.2.5.2 Subcase B.2: For $c_2 - c_3 = 0$,

$F(u) = k e^{\frac{2c_2}{c_4} u}$.

This is equivalent to $F(u) = e^u$ under equivalence transformations (28).

Obviously, Case B corresponds to the solutions set of $IS_3 = 0$.

Solving the equations of $DCS_1 = 0$, we have

$$\text{Zero}(DCS_1, IS_1/IS_3 * IS_4) = \left\{ \begin{array}{l} \xi = c_2 x + c_1 v + c_5, \tau = c_1 u + c_6, \\ \eta = 2(c_1 u + c_2) F(u)/F'(u), \end{array} \text{ with } c_1 \neq 0. \right\} \tag{30}$$

This is a unifying expression for the corresponding symmetries of Subcases A.1, A.2 and A.3.

Repeating our algorithm for $c_1 = 0$, or equivalently for $F(u) = u^\alpha$, $F(u) = e^u$ under (28), we have

$$\begin{aligned} &\text{Zero}(DPS, IS_3/IS_1) \cup \text{Zero}(DPS, IS_4/IS_1) \\ &\quad = \text{Zero}(DCS_2) \cup \text{Zero}(DCS_3) \cup \text{Zero}(DCS_4), \end{aligned} \tag{31}$$

with $F(u) = e^u$ for DCS_2 and $F(u) = u^\alpha$, $\alpha \neq -2, -4/3$ for DCS_3 and $F(u) = u^{-4/3}$ for DCS_4;

$$\text{Zero}(DPS, IS_2) = \text{Zero}(DCS_5), \tag{32}$$

with $F(u) = u^{-2}$.

Combining (26), (27), (30–32) and the equivalence transformation (28), we get the decomposition (12) for the DPS.

Here

$$DCS_2 = \{\eta_x, \eta_t, \eta_u, \eta_v, \tau_x, \tau_t, \tau_u, \tau_v, \xi_t, \xi_u, \xi_v, \phi_x - \eta, 2\xi_x - \eta, 2\phi_v - \eta, \},$$

$$DCS_3 = \{\tau_x, \tau_t, \tau_u, \tau_v, \eta_x, \eta_t, \eta_v, \phi_x, \phi_t, \phi_u, u\eta_u - \eta, 2u\phi_v - (\alpha + 2)\eta, 2u\xi_x - \alpha\eta\},$$

$$DCS_4 = \{\tau_x, \tau_t, \tau_u, \tau_v, \xi_t, \xi_u, \xi_v, \eta_x, \eta_t, \eta_v, \phi_x, \phi_t, \phi_u, u\eta_u - \eta, 3u\phi_v - \eta, 3u\xi_x + 2\eta\},$$

$$DCS_5 = \left\{ \begin{array}{l} \phi_x, \phi_v, \xi_{xu}, \xi_{xt}, \xi_{xx}, \eta - u\eta_u - u^2\xi_v, \\ \phi_u - 2u\xi_u + 2\xi_t, \phi_t - 2u^2\xi_u + 2u\xi_t, \eta_v - u^3\xi_u + 2u^2\xi_t, \\ \eta_x - u^4\xi_u + u^3\xi_t, u\xi_{uv} - \xi_{tv}, \xi_{vv} - \xi_t - \xi_{tt}, \\ 2u\xi_u + u^2\xi_{uu} - \xi_t - \xi_{tt}, u\xi_{tu} - \xi_{tt}, \\ \eta + \eta_t + u^2\xi_v + u\xi_x, u^2\xi_u - u\xi_t + \xi_{xv}, u^2\xi_u - \tau_v, \xi_v - \tau_u, \\ \eta + u^2\xi_v + u\xi_x - u\tau_t, u^3\xi_u - u^2\xi_t + \tau_x. \end{array} \right\}$$

and

$$\text{Zero}(DCS_2) = \{\xi = c_1 x + c_3, \tau = c_4, \eta = 2c_1, \phi = c_1 v + 2c_1 x + c_2\},$$

$$\text{Zero}(DCS_3) = \left\{\xi = \frac{\alpha}{2} c_2 x + c_4, \tau = c_1, \eta = c_2 u, \phi = \frac{2 + \alpha}{2} c_2 v + c_3\right\},$$

$$\text{Zero}(DCS_4) = \left\{\xi = -\frac{2}{3} c_2 x + c_4, \tau = c_1, \eta = c_2 u, \phi = \frac{1}{3} c_2 v + c_3\right\},$$

$$\text{Zero}(DCS_5) = \left\{ \begin{array}{l} \xi = \frac{c_1}{u} + (c_2 + c_1 v)x + B\left(v, e^t u\right), \\ \tau = c_4 + c_1(ux - v) + e^{-t} A\left(v, e^t u\right), \\ \eta = -\left(c_2 + c_1(v + ux) + e^{-t} A\left(v, e^t u\right)\right) u, \\ \phi = c_3 - 2c_1(t + \log u), \end{array} \right\}$$

with $A_V(V, U) = U^2 B_U(V, U)$, $B_V(V, U) = A_U(V, U)$, and $U = e^t u$, $V = v$.

The cases $\text{Zero}(DCS_i)$, $i = 1, 2, \ldots, 5$ and corresponding classifying equations $IS_i = 0 (i = 1, 2, 3, 4)$ yield a complete potential symmetry classification of the original wave equation, in which DCS_1 and DCS_5 efficiently extend the classical symmetries of the wave equations with the classifying equation $IS_1 = 0$. Its solutions are given in Case A and Case B. The DCS_5 corresponds to a linearizable case.

Remark Of course, the equivalence transformations (28) are not complete. For example, the PDE system (25) has additional equivalence transformation $x^* = \epsilon v + x$, $t^* = \log(\epsilon u + 1) + t$, $u^* = u/(1+\epsilon u)$, $v^* = v$, $F^* = (1+\epsilon u)^2 F$ with generator $v\partial_x + u\partial_t - u^2\partial_u + 2uF\partial_F$. Determining the set of all equivalence transformations is equivalent to determining the classical symmetries of the extended system consisting of the given PDEs with auxiliary equations involving the parameter, which are set up by representing the parameter as being independent of some variables in $X \times \partial U$ (see details in [24,27]). Additional equivalence transformations may help to further simplify the presentation of the symmetry classification (see an example in [33]), but this is not a focus of the present paper.

4.3 Nonclassical symmetries of Burgers' equation with a parameter

In the following we determine nonclassical symmetries of Burgers' equation $u_t + H(u)u_x^2 + u_{xx} = 0$ with a parameter $\theta = H(u) \neq 0$.

Suppose the corresponding InfV is $\mathcal{X} = \partial_t + \xi(x,t,u)\partial_x + \eta(x,t,u)\partial_u$. Thus the DTEs DPS=0 are given by

$$\begin{aligned} \text{DPS} = \{&H(u)\xi_u - \xi_{uu}, H(u)\eta_u + \eta_{uu} - 2\xi\xi_u - 2\xi_{xu} + \eta H'(u), \\ &\eta_t + \eta_{xx} + 2\eta\xi_x, 2H(u)\eta_x + 2\eta_{xu} + 2\eta\xi_u - \xi_t - 2\xi\xi_x - \xi_{xx}\}. \end{aligned}$$

This is a nonlinear dps. Under the basic rank $x \prec t \prec u \prec \xi \prec \eta$ and by Algorithm B, we obtain the dchar-set of the dps given by

$$\begin{aligned} \text{DCS} = \{&\xi_u, \eta_t + \eta_{xx} + 2\eta\xi_x, 2H(u)\eta_x + 2\eta_{xu} - \xi_t - 2\xi\xi_x - \xi_{xx}, H(u)\eta_u + \eta_{uu} + \eta H'(u), \\ &2H(u)\eta_t + 2\eta_{tu} + 4H(u)\eta\xi_x + 4\eta_u\xi_x + 2\xi_x^2 + \xi_{xt} + 2\xi\xi_{xx} + \xi_{xxx}\}, \end{aligned}$$

for any value of the parameter $\theta = H(u)$ (no classifying equations exist) and

$$\text{Zero(DPS)=Zero(DCS)} = \{\xi = \gamma(x,t), \eta = \mathrm{e}^{-\int H(u)\mathrm{d}u}(\alpha(x,t)\int \mathrm{e}^{\int H(u)\mathrm{d}u}\mathrm{d}u + \beta(x,t))\},$$

where $\alpha(x,t)$, $\beta(x,t)$, $\gamma(x,t)$ satisfy

$$\begin{aligned} &\alpha_t(x,t) + \alpha_{xx}(x,t) + 2\alpha(x,t)\gamma_x(x,t) = 0, \\ &\beta_t(x,t) + \beta_{xx}(x,t) + 2\beta(x,t)\gamma_x(x,t) = 0, \\ &\gamma_t(x,t) + \gamma_{xx}(x,t) + 2\gamma(x,t)\gamma_x(x,t) = 2\alpha_x(x,t). \end{aligned} \tag{33}$$

This shows that Burgers' equation admits a very rich set of nonclassical symmetries in terms of the unified expression given in Zero(DCS).

Similarly we get the classical symmetries of Burgers' equation with InfV $\mathcal{X} = \xi(x,t,u)\partial_x + \tau(x,t,u)\partial_t + \phi(x,t,u)\partial_u$. Here

$$\begin{aligned} &\xi = (c_1x + c_4)t + \frac{1}{2}c_2x + c_5, \tau = c_1t^2 + c_2t + c_3, \\ &\eta = \mathrm{e}^{-\int H(u)\mathrm{d}u}\left\{\left(\frac{1}{2}(\frac{1}{2}c_1(x^2 - t) + c_4x) + c_6\right)\int \mathrm{e}^{\int H(u)\mathrm{d}u}du + \gamma(x,t)\right\} \end{aligned}$$

for an arbitrary value of $H(u)$ and $\gamma = \gamma(x,t)$ satisfies $\gamma_t + \gamma_{xx} = 0$.

Remark Actually, the structure of the classical symmetries of Burgers' equation shows that it can be linearized to $\gamma_t + \gamma_{xx} = 0$ which does not contain any parameter [2, Chap. 6]. So in classical and non-classical symmetry classifications no splitting cases arise. Here, we just emphasize the application of our algorithm to a nonlinear system.

5 Concluding remarks and some discussion

By using the dchar-set algorithm and the zero decomposition theorem given in Wu's method (Theorem 3), we obtain the complete decomposition of the solution set of the DTEs in terms of a series of zero sets of dchar-sets with all classifying equations in (12). Based on the decomposition, we get an alternative algorithm for the complete symmetry classification of a parametric family of PDEs. The classification in the algorithm is given by the correspondence between each branch in the decomposition and a class of symmetries with associated classifying equations. The well-ordered equations corresponding to dchar-sets in the zero decomposition are usually solved more easily than the original one or provide a real simplification to determine an explicit symmetry classification.

Interesting and new aspects of our algorithm are as follows. This is a unified and systematic algorithm for more general symmetry classifications of PDEs. The algorithm differs from existing algorithms by using a fundamentally different theory and algorithm. The algorithm can also be easily used to solve other differential problems, such as integrating factors, first integrals and conservation laws (classifications) of PDEs [47–50]. Particularly, from the dchar-set of DTEs for symmetries, we may obtain more information about symmetries, such as symmetry dimension, symmetry structure, Taylor-series solutions of the DTEs, etc [4–8]. As far as the authors know, this is the first paper in which the complete symmetry classifications of PDEs are systematically investigated by Wu's method.

In order to compare with related works, let us describe in a few words some existing algorithms.

In [8,9], Schwarz described and implemented the algorithm *involutuionSystem* (package `SYMSIZE`), based on the theory of differential equations of Riquier and Janet [19,20], to transform a linear system of PDEs into involutive form. The algorithm can determine the size of the Lie symmetry group of a given system of PDEs without having to integrate the corresponding DTEs.

In [6,7], Reid et al designed *rif*-algorithms which use a finite number of differentiations and algebraic operations to simplify an analytic (nonlinear) system of PDEs to what they call 'a reduced involutions form'(*rif-form*), which includes the integrability conditions of the system and satisfies a constant rank condition. An integration-free algorithm based on the reduced form and formal power series analysis is developed to determine Lie symmetries of PDEs and many other applications. The implemented CAS package `rifsimp` of the algorithm is available in Maple.

In [11,12], Mansfield designed an algorithm to compute differential Gröbner bases for polynomial PDEs. Fundamental tools in the algorithm are the Kolchin–Ritt algorithm, a differential analogue of Buchberger's algorithm with pseduo-reduction instead of reduction, and the `diffgrob2` package is implemented in Maple, which is designed to calculate the elimination ideals, integrability conditions and compatibility conditions of a system of nonlinear partial differential equations.

In [14,15], Boulier et al proposed Rosenfeld–Gröbner algorithms which borrow from the algorithms of Seidenberg [51] the idea to combine Hilbert's theorem on zeros and Ritt's reduction algorithm. They decide the membership problem in the radical ideal generated by finite differential polynomials set by successively eliminating all the unknowns appearing in the polynomial of the set. The implemented package `diffalg` based on the algorithm is available in Maple. Many of these algorithms were motivated by symmetry analysis.

In [36, Chap. 3], [37], Wu developed a dchar-set method on the basis of the previous work of Ritt [22] for the purpose of theorem proving. It is much different in theoretical and algorithmic aspects from its roots and those mentioned above. The concept of dchar-sets is a key element in both Ritt's and Wu's theory. The concept of a dchar-set is defined for an algebraic ideal generated by a dps in Ritt's work, while the dchar-set in Wu's method is simply defined for the dps (bases of ideals). The main focus of Wu's method is to directly deal with the zero set of a dps. This method decomposes the zero set Zero(P) of a dps P into finitely many zero sets Zero($\mathrm{DCS}_i/\mathrm{IS}_i$) of the dchar-sets DCS_i of the dps P. As a result, a fundamental relation provided by the dchar-set algorithm is a decomposition (8), which is called a weak form decomposition.

To compute zero decompositions of finer form, the dchar-set method proceeds further by imposing an irreducibility requirement and computing irreducible dchar-sets. This process provides us with similarity relations (8) where the DCS_i are irreducible. This is called a strong-form decomposition. In practice, it is more efficient to compute weak forms than strong forms. A weak form is also very useful, as its components possess many interesting algebraic properties [22, Chap. 5], [36, Chap. 3], [37]. Undoubtedly, there are other ways to achieve (8) besides the algorithms in Wu [36, Chap. 3], [37]. For example, d-chain-sets here may be replaced by coherent autoreduced sets in Rosenfeld [52]; elimination techniques in Seidenberg [51] may be combined with the dchar-set method. In [53], a modification of Wu's decomposition (8) is given by replacing the dchar-sets by so called differential triangulation (d-tri) sets with coherent conditions instead of integrability conditions, which significantly reduces the computation efforts. In theorem proving, the concepts of differential-algebra ideals are used. This indicates at some level the relationship between Wu's method and algebra ideals. Wu's method was developed originally for mechanical theorem proving.

All of these algorithms have in commons the idea of reduction. A consistent theme in these algorithms is the idea that one first reduces the PDEs under consideration to some kind of 'standard form' (e.g. *Gröbner basis*, *dchar-set*, *rif-form*, *involutive form*, etc), then integrates or uses the obtained 'well ordering' system. The reduction operations used in these algorithms have origins in the works of Janet–Riquier [20], Seidenberg [51], Ritt [22, Chap. 5], etc. However, the procedures for obtaining the reduced 'standard form' are different.

Our classification algorithms in this paper are stated by utilizing only the weak form of zero decomposition results (8) and Wu's algorithm for dchar-sets of a dps. Assuredly, if we use the modified algorithm in [53], we will get a refinement of the classification algorithm by following the same idea in the paper. Moreover, the other methods mentioned above can be used for solving symmetry-classification problems [34]. Here, we just give an alternative method for solving symmetry-classification problems.

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